On the Magic Matrix by Makhlin and the B-C-H Formula in $SO(4)$

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Abstract

A closed expression to the Baker–Campbell–Hausdorff (B-C-H) formula in $SO(4)$ is given by making use of the magic matrix by Makhlin. As far as we know this is the **first nontrivial example** on (semi–) simple Lie groups summing up all terms in the B-C-H expansion.

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1 Introduction

The Baker–Campbell–Hausdorff (B–C–H) formula is one of fundamental ones in elementary Linear Algebra (or Lie group). That is, we have

$$\mathrm{e}^{A}\mathrm{e}^{B} = \mathrm{e}^{BCH(A,B)}$$

where A and B are elements in some algebra and

$$BCH(A,B) = A + B + \frac{1}{2}[A,B] + \frac{1}{12}\left\{[[A,B],B] + [A,[A,B]]\right\} + \cdots.$$

See for example [1] or [2]. For simplicity we call this the B–C–H expansion in the text. Although the formula is "elementary", it is difficult (almost impossible ?) to sum up all terms in $BCH(A,B)$. If $[A,B]$ and A, B commute, then we have

$$BCH(A,B) = A + B + \frac{1}{2}[A,B].$$

However, this is very useful but exceptional, [3].

By the way, it is not difficult to give a close expression to $BCH(A,B)$ for the case of $SU(2)$ because it is easy to treat. We would like to use this expression in the paper.

Next, to apply the Makhlin's theorem to the problem let us explain it. The isomorphism

$$SU(2) \otimes SU(2) \cong SO(4)$$

is one of well–known theorems in elementary representation theory and is a typical characteristic of four dimensional Euclidean space. In [4] Makhlin gave it **the adjoint expression** like

$$F : SU(2) \otimes SU(2) \longrightarrow SO(4), \quad F(A \otimes B) = Q^{\dagger}(A \otimes B)Q$$

with some unitary matrix $Q \in U(4)$. As far as we know this is the first that the map was given by the adjoint action. See also [5], where a bit different matrix R has been used in place of Q because Q is of course not unique. This Q (R in our notation) is interesting enough and is called the magic matrix by Makhlin, see [6] and [7].

Since a close expression for the B–C–H formula in the case of $SU(2)$ is known, we can also obtain the close expression for $BCH(A,B)$ in the case of $SO(4)$

$$\mathrm{e}^{A}\mathrm{e}^{B} = \mathrm{e}^{BCH(A,B)} \quad \text{for} \quad A, B \in so(4)$$

by making use of the magic matrix by Makhlin. This is the main result in the paper. As far as we know this is the first nontrivial example on (semi–) simple Lie groups summing up all terms in the B-C-H expansion.

2 Review on the Magic Matrix

In this section we review the result in [5] within our necessity, which is a bit different from the one in [4].

The 1–qubit space is $\mathbf{C}^2 = \mathrm{Vect}_{\mathbf{C}}\{|0\rangle, |1\rangle\}$ where

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1}$$

Let $\{\sigma_1, \sigma_2, \sigma_3\}$ be the Pauli matrices acting on the space

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

Next let us consider the 2–qubit space. Now we use notations on tensor product which are different from usual ones. That is,

$$\mathbf{C}^2 \otimes \mathbf{C}^2 = \{a \otimes b \mid a, b \in \mathbf{C}^2\},$$

while

$$\mathbf{C}^2 \widehat{\otimes} \mathbf{C}^2 = \left\{ \sum_{j=1}^{k} c_j a_j \otimes b_j \mid a_j, b_j \in \mathbf{C}^2,\ c_j \in \mathbf{C},\ k \in \mathbf{N} \right\} \cong \mathbf{C}^4.$$

Then

$$\mathbf{C}^2 \widehat{\otimes} \mathbf{C}^2 = \mathrm{Vect}_{\mathbf{C}}\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$$

where $|ab\rangle = |a\rangle \otimes |b\rangle$ $(a, b \in \{0, 1\})$.

By $H_0(2;\mathbf{C})$ we show the set of all traceless hermite matrices in $M(2;\mathbf{C})$. Then it is well–known

$$H_0(2;\mathbf{C}) = \{a \equiv a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 \mid a_1, a_2, a_3 \in \mathbf{R}\}$$

and $H_0(2;\mathbf{C}) \cong su(2)$ where $su(2)$ is the Lie algebra of the group $SU(2)$.

By making use of the Bell bases $\{|\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle, |\Psi_4\rangle\}$ defined by

$$\begin{aligned} |\Psi_1\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad |\Psi_2\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \\ |\Psi_3\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \quad |\Psi_4\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \end{aligned} \tag{3}$$

we can give the isomorphism as the adjoint action (the Makhlin's theorem) as follows

$$F : SU(2) \otimes SU(2) \longrightarrow SO(4), \quad F(A \otimes B) = R^\dagger (A \otimes B) R$$

where

$$R = (|\Psi_1\rangle, -i|\Psi_2\rangle, -|\Psi_3\rangle, -i|\Psi_4\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & -i & -1 & 0 \\ 0 & -i & 1 & 0 \\ 1 & 0 & 0 & i \end{pmatrix}. \tag{4}$$

Note that the unitary matrix R is a bit different from Q in [4].

Let us consider this problem in a Lie algebra level because it is in general not easy to treat it in a Lie group level.

$$\begin{tikzcd} \mathfrak{L}(SU(2) \otimes SU(2)) \arrow[r, "f"] \arrow[d, "\exp"'] & \mathfrak{L}(SO(4)) \arrow[d, "\exp"] \\ SU(2) \otimes SU(2) \arrow[r, "F"'] & SO(4) \end{tikzcd}$$

Since the Lie algebra of $SU(2) \otimes SU(2)$ is

$$\mathfrak{L}(SU(2) \otimes SU(2)) = \{i(a \otimes 1_2 + 1_2 \otimes b) \mid a, b \in H_0(2;\mathbf{C})\},$$

we have only to examine

$$f(i(a \otimes 1_2 + 1_2 \otimes b)) = iR^\dagger (a \otimes 1_2 + 1_2 \otimes b)R \in \mathfrak{L}(SO(4)) \equiv so(4). \tag{5}$$

If we set $a = \sum_{j=1}^{3} a_j \sigma_j$ and $b = \sum_{j=1}^{3} b_j \sigma_j$ then the right hand side of (5) becomes

$$iR^\dagger (a \otimes 1_2 + 1_2 \otimes b)R = \begin{pmatrix} 0 & a_1 + b_1 & a_2 - b_2 & a_3 + b_3 \\ -(a_1 + b_1) & 0 & a_3 - b_3 & -(a_2 + b_2) \\ -(a_2 - b_2) & -(a_3 - b_3) & 0 & a_1 - b_1 \\ -(a_3 + b_3) & a_2 + b_2 & -(a_1 - b_1) & 0 \end{pmatrix}. \tag{6}$$

Conversely, if

$$A = \begin{pmatrix} 0 & f_{12} & f_{13} & f_{14} \\ -f_{12} & 0 & f_{23} & f_{24} \\ -f_{13} & -f_{23} & 0 & f_{34} \\ -f_{14} & -f_{24} & -f_{34} & 0 \end{pmatrix} \in so(4)$$

then we obtain

$$RAR^\dagger = i(a \otimes 1_2 + 1_2 \otimes b) \tag{7}$$

with

$$a = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 = \frac{f_{12} + f_{34}}{2}\sigma_1 + \frac{f_{13} - f_{24}}{2}\sigma_2 + \frac{f_{14} + f_{23}}{2}\sigma_3, \tag{8}$$

$$b = b_1\sigma_1 + b_2\sigma_2 + b_3\sigma_3 = \frac{f_{12} - f_{34}}{2}\sigma_1 - \frac{f_{13} + f_{24}}{2}\sigma_2 + \frac{f_{14} - f_{23}}{2}\sigma_3. \tag{9}$$

It is very interesting to note that a is the self–dual part and b the anti–self–dual one.

The matrix R is called the magic one by Makhlin. Readers will understand why this is called magic through this paper.

3 B-C-H Formula for SU(2)

In this section we give a closed expression to the B–C–H formula for $SU(2)$, which is more or less well–known. See for example [8] and [9].

First of all let us recall the well–known formula.

$$\mathrm{e}^{i(x\sigma_1+y\sigma_2+z\sigma_3)} = \cos r\mathbf{1} + \frac{\sin r}{r} i(x\sigma_1 + y\sigma_2 + z\sigma_3) \tag{10}$$

where $r = \sqrt{x^2+y^2+z^2}$ and $\mathbf{1} = 1_2$ for simplicity. This is a simple exercise.

For the group $SU(2)$ it is easy to sum up all terms in the B–C–H expansion by making use of the above one. Before stating the result let us prepare some notations. For

$$X = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3, \quad Y = y_1\sigma_1 + y_2\sigma_2 + y_3\sigma_3 \in H_0(2, \mathbf{C})$$

we set

$$X \longrightarrow \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad Y \longrightarrow \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

and

$$\mathbf{x}\cdot\mathbf{y} = x_1y_1 + x_2y_2 + x_3y_3, \quad |\mathbf{x}| = \sqrt{\mathbf{x}\cdot\mathbf{x}}, \quad |\mathbf{y}| = \sqrt{\mathbf{y}\cdot\mathbf{y}}$$

and

$$-\frac{i}{2}[X, Y] \longrightarrow \mathbf{x}\times\mathbf{y} = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}.$$

Now we are in a position to state the B–C–H formula for $SU(2)$.

$$\mathrm{e}^{iX}\mathrm{e}^{iY} = \mathrm{e}^{iZ} \ : \quad Z = \alpha X + \beta Y + \gamma\frac{i}{2}[X, Y] \tag{11}$$

where

$$\begin{aligned} &\alpha \equiv \alpha(\mathbf{x},\mathbf{y}) = \frac{\sin^{-1}\rho}{\rho}\frac{\sin|\mathbf{x}|\cos|\mathbf{y}|}{|\mathbf{x}|}, \quad \beta \equiv \beta(\mathbf{x},\mathbf{y}) = \frac{\sin^{-1}\rho}{\rho}\frac{\cos|\mathbf{x}|\sin|\mathbf{y}|}{|\mathbf{y}|}, \\ &\gamma \equiv \gamma(\mathbf{x},\mathbf{y}) = \frac{\sin^{-1}\rho}{\rho}\frac{\sin|\mathbf{x}|\sin|\mathbf{y}|}{|\mathbf{x}||\mathbf{y}|} \end{aligned} \tag{12}$$

with

$$\begin{aligned} &\rho^2 \equiv \rho(\mathbf{x},\mathbf{y})^2 \\ &= \sin^2|\mathbf{x}|\cos^2|\mathbf{y}| + \sin^2|\mathbf{y}| - \frac{\sin^2|\mathbf{x}|\sin^2|\mathbf{y}|}{|\mathbf{x}|^2|\mathbf{y}|^2}(\mathbf{x}\cdot\mathbf{y})^2 + \frac{2\sin|\mathbf{x}|\cos|\mathbf{x}|\sin|\mathbf{y}|\cos|\mathbf{y}|}{|\mathbf{x}||\mathbf{y}|}(\mathbf{x}\cdot\mathbf{y}). \end{aligned}$$

The proof is not difficult, so is left to readers.

Some comments are in order.

(1) In [8] the Euler angle's parametrisation is used. However, it is particular to the case of $SU(2) \cong S^3$ and there is no generality, so we don't use it in the paper.

(2) From our result it is easy to see the result in [9] by using the adjoint representation $Ad : SU(2) \longrightarrow SO(3)$ (see for example [5]).

4 B-C-H Formula for SO(4)

In this section we also give a closed expression to the B–C–H formula for $SO(4)$ by use of the results in the preceding two sections. Before that let us prepare some notations for simplicity.

For $A, B \in so(4)$

$$A = \begin{pmatrix} 0 & f_{12} & f_{13} & f_{14} \\ -f_{12} & 0 & f_{23} & f_{24} \\ -f_{13} & -f_{23} & 0 & f_{34} \\ -f_{14} & -f_{24} & -f_{34} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & g_{12} & g_{13} & g_{14} \\ -g_{12} & 0 & g_{23} & g_{24} \\ -g_{13} & -g_{23} & 0 & g_{34} \\ -g_{14} & -g_{24} & -g_{34} & 0 \end{pmatrix} \tag{13}$$

we can set

$$RAR^\dagger = i(\mathbf{a}_1 \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{a}_2), \quad RBR^\dagger = i(\mathbf{b}_1 \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{b}_2)$$

and

$$\mathbf{a}_1 = \frac{f_{12}+f_{34}}{2}\sigma_1 + \frac{f_{13}-f_{24}}{2}\sigma_2 + \frac{f_{14}+f_{23}}{2}\sigma_3, \ \mathbf{a}_2 = \frac{f_{12}-f_{34}}{2}\sigma_1 - \frac{f_{13}+f_{24}}{2}\sigma_2 + \frac{f_{14}-f_{23}}{2}\sigma_3,$$
$$\mathbf{b}_1 = \frac{g_{12}+g_{34}}{2}\sigma_1 + \frac{g_{13}-g_{24}}{2}\sigma_2 + \frac{g_{14}+g_{23}}{2}\sigma_3, \ \mathbf{b}_2 = \frac{g_{12}-g_{34}}{2}\sigma_1 - \frac{g_{13}+g_{24}}{2}\sigma_2 + \frac{g_{14}-g_{23}}{2}\sigma_3$$

and

$$\vec{\mathbf{a}}_1 = \begin{pmatrix} \frac{f_{12}+f_{34}}{2} \\ \frac{f_{13}-f_{24}}{2} \\ \frac{f_{14}+f_{23}}{2} \end{pmatrix}, \quad \vec{\mathbf{a}}_2 = \begin{pmatrix} \frac{f_{12}-f_{34}}{2} \\ -\frac{f_{13}+f_{24}}{2} \\ \frac{f_{14}-f_{23}}{2} \end{pmatrix}; \quad \vec{\mathbf{b}}_1 = \begin{pmatrix} \frac{g_{12}+g_{34}}{2} \\ \frac{g_{13}-g_{24}}{2} \\ \frac{g_{14}+g_{23}}{2} \end{pmatrix}, \quad \vec{\mathbf{b}}_2 = \begin{pmatrix} \frac{g_{12}-g_{34}}{2} \\ -\frac{g_{13}+g_{24}}{2} \\ \frac{g_{14}-g_{23}}{2} \end{pmatrix},$$

and

$$\alpha_1 = \alpha(\vec{\mathbf{a}}_1, \vec{\mathbf{b}}_1), \quad \beta_1 = \beta(\vec{\mathbf{a}}_1, \vec{\mathbf{b}}_1), \quad \gamma_1 = \gamma(\vec{\mathbf{a}}_1, \vec{\mathbf{b}}_1),$$
$$\alpha_2 = \alpha(\vec{\mathbf{a}}_2, \vec{\mathbf{b}}_2), \quad \beta_2 = \beta(\vec{\mathbf{a}}_2, \vec{\mathbf{b}}_2), \quad \gamma_2 = \gamma(\vec{\mathbf{a}}_2, \vec{\mathbf{b}}_2).$$

See the preceding two sections. Then we have

$$\begin{aligned}
\mathrm{e}^A\mathrm{e}^B &= R^\dagger R\mathrm{e}^A R^\dagger R\mathrm{e}^B R^\dagger R \\
&= R^\dagger \mathrm{e}^{RAR^\dagger}\mathrm{e}^{RBR^\dagger} R \\
&= R^\dagger \mathrm{e}^{i(\mathbf{a}_1\otimes\mathbf{1}+\mathbf{1}\otimes\mathbf{a}_2)}\mathrm{e}^{i(\mathbf{b}_1\otimes\mathbf{1}+\mathbf{1}\otimes\mathbf{b}_2)} R \\
&= R^\dagger \left(\mathrm{e}^{i\mathbf{a}_1}\otimes\mathrm{e}^{i\mathbf{a}_2}\right)\left(\mathrm{e}^{i\mathbf{b}_1}\otimes\mathrm{e}^{i\mathbf{b}_2}\right) R \\
&= R^\dagger \left(\mathrm{e}^{i\mathbf{a}_1}\mathrm{e}^{i\mathbf{b}_1}\right)\otimes\left(\mathrm{e}^{i\mathbf{a}_2}\mathrm{e}^{i\mathbf{b}_2}\right) R \\
&= R^\dagger \mathrm{e}^{i(\alpha_1\mathbf{a}_1+\beta_1\mathbf{b}_1+\gamma_1\frac{i}{2}[\mathbf{a}_1,\mathbf{b}_1])}\otimes\mathrm{e}^{i(\alpha_2\mathbf{a}_2+\beta_2\mathbf{b}_2+\gamma_2\frac{i}{2}[\mathbf{a}_2,\mathbf{b}_2])} R \\
&= R^\dagger \mathrm{e}^{i\left\{(\alpha_1\mathbf{a}_1+\beta_1\mathbf{b}_1+\gamma_1\frac{i}{2}[\mathbf{a}_1,\mathbf{b}_1])\otimes\mathbf{1}+\mathbf{1}\otimes(\alpha_2\mathbf{a}_2+\beta_2\mathbf{b}_2+\gamma_2\frac{i}{2}[\mathbf{a}_2,\mathbf{b}_2])\right\}} R \\
&= \mathrm{e}^{iR^\dagger\left\{(\alpha_1\mathbf{a}_1+\beta_1\mathbf{b}_1+\gamma_1\frac{i}{2}[\mathbf{a}_1,\mathbf{b}_1])\otimes\mathbf{1}+\mathbf{1}\otimes(\alpha_2\mathbf{a}_2+\beta_2\mathbf{b}_2+\gamma_2\frac{i}{2}[\mathbf{a}_2,\mathbf{b}_2])\right\}R} \\
&= \mathrm{e}^{BCH(A,B)}
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
BCH(A,B) &= iR^\dagger\left\{\left(\alpha_1\mathbf{a}_1+\beta_1\mathbf{b}_1+\gamma_1\frac{i}{2}[\mathbf{a}_1,\mathbf{b}_1]\right)\otimes\mathbf{1}+\mathbf{1}\otimes\left(\alpha_2\mathbf{a}_2+\beta_2\mathbf{b}_2+\gamma_2\frac{i}{2}[\mathbf{a}_2,\mathbf{b}_2]\right)\right\}R \\
&= \begin{pmatrix} 0 & (12) & (13) & (14) \\ -(12) & 0 & (23) & (24) \\ -(13) & -(23) & 0 & (34) \\ -(14) & -(24) & -(34) & 0 \end{pmatrix}
\end{aligned} \tag{15}$$

whose entries are

$$\begin{aligned}
(12) = {} & \alpha_1\frac{f_{12}+f_{34}}{2}+\beta_1\frac{g_{12}+g_{34}}{2}-\gamma_1\left(\frac{f_{13}-f_{24}}{2}\frac{g_{14}+g_{23}}{2}-\frac{f_{14}+f_{23}}{2}\frac{g_{13}-g_{24}}{2}\right)+ \\
& \alpha_2\frac{f_{12}-f_{34}}{2}+\beta_2\frac{g_{12}-g_{34}}{2}-\gamma_2\left(-\frac{f_{13}+f_{24}}{2}\frac{g_{14}-g_{23}}{2}+\frac{f_{14}-f_{23}}{2}\frac{g_{13}+g_{24}}{2}\right), \\
(13) = {} & \alpha_1\frac{f_{13}-f_{24}}{2}+\beta_1\frac{g_{13}-g_{24}}{2}-\gamma_1\left(\frac{f_{14}+f_{23}}{2}\frac{g_{12}+g_{34}}{2}-\frac{f_{12}+f_{34}}{2}\frac{g_{14}+g_{23}}{2}\right)+
\end{aligned}$$

$$\alpha_2\frac{f_{13}+f_{24}}{2}+\beta_2\frac{g_{13}+g_{24}}{2}-\gamma_2\left(-\frac{f_{14}-f_{23}}{2}\frac{g_{12}-g_{34}}{2}+\frac{f_{12}-f_{34}}{2}\frac{g_{14}-g_{23}}{2}\right),$$

$$(14)=\alpha_1\frac{f_{14}+f_{23}}{2}+\beta_1\frac{g_{14}+g_{23}}{2}-\gamma_1\left(\frac{f_{12}+f_{34}}{2}\frac{g_{13}-g_{24}}{2}-\frac{f_{13}-f_{24}}{2}\frac{g_{12}+g_{34}}{2}\right)+$$

$$\alpha_2\frac{f_{14}-f_{23}}{2}+\beta_2\frac{g_{14}-g_{23}}{2}-\gamma_2\left(-\frac{f_{12}-f_{34}}{2}\frac{g_{13}+g_{24}}{2}+\frac{f_{13}+f_{24}}{2}\frac{g_{12}-g_{34}}{2}\right),$$

$$(23)=\alpha_1\frac{f_{14}+f_{23}}{2}+\beta_1\frac{g_{14}+g_{23}}{2}-\gamma_1\left(\frac{f_{12}+f_{34}}{2}\frac{g_{13}-g_{24}}{2}-\frac{f_{13}-f_{24}}{2}\frac{g_{12}+g_{34}}{2}\right)$$

$$-\alpha_2\frac{f_{14}-f_{23}}{2}-\beta_2\frac{g_{14}-g_{23}}{2}+\gamma_2\left(-\frac{f_{12}-f_{34}}{2}\frac{g_{13}+g_{24}}{2}+\frac{f_{13}+f_{24}}{2}\frac{g_{12}-g_{34}}{2}\right),$$

$$(24)=-\alpha_1\frac{f_{13}-f_{24}}{2}-\beta_1\frac{g_{13}-g_{24}}{2}+\gamma_1\left(\frac{f_{14}+f_{23}}{2}\frac{g_{12}+g_{34}}{2}-\frac{f_{12}+f_{34}}{2}\frac{g_{14}+g_{23}}{2}\right)+$$

$$\alpha_2\frac{f_{13}+f_{24}}{2}+\beta_2\frac{g_{13}+g_{24}}{2}-\gamma_2\left(-\frac{f_{14}-f_{23}}{2}\frac{g_{12}-g_{34}}{2}+\frac{f_{12}-f_{34}}{2}\frac{g_{14}-g_{23}}{2}\right),$$

$$(34)=\alpha_1\frac{f_{12}+f_{34}}{2}+\beta_1\frac{g_{12}+g_{34}}{2}-\gamma_1\left(\frac{f_{13}-f_{24}}{2}\frac{g_{14}+g_{23}}{2}-\frac{f_{14}+f_{23}}{2}\frac{g_{13}-g_{24}}{2}\right)$$

$$-\alpha_2\frac{f_{12}-f_{34}}{2}-\beta_2\frac{g_{12}-g_{34}}{2}+\gamma_2\left(-\frac{f_{13}+f_{24}}{2}\frac{g_{14}-g_{23}}{2}+\frac{f_{14}-f_{23}}{2}\frac{g_{13}+g_{24}}{2}\right).$$

We could obtain the closed expression for the B–C–H formula and this is our main result in the paper. Note that we can transform (15) into various forms, which will be left to readers. As far as we know this is the first nontrivial example summing up all terms in the B–C–H expansion.

5 Discussion

In this letter we studied the B–C–H formula for the case of $SO(4)$ and obtained the closed expression by making use of the formula for the case of $SU(2)$ and the magic matrix R by Makhlin.

The Makhlin's matrix is essential in the case of $SO(4)$ and the readers should recognize the reason why it is called magic. It will be used in Quantum Computation and Mathematical Physics moreover, see for example [10], [11].

Last, we would like to make a comment on some generalization of our work. In [8] an interesting method to calculate the B–C–H formula for the case of $SU(n)$ has been presented.

However, to perform it explicitly may be difficult even for the case $SU(4)$ [1]

$$e^{iX}e^{iY} = e^{iBCH(X,Y)} \quad \text{for} \quad X, Y \in H_0(4, \mathbf{C}).$$

To apply our method to the same case may be useful, which will be reported elsewhere.

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